Section 8.2

Vertex Colorings

COLORING OF A GRAPH

An assignment of colors to the vertices of a graph G (one color per vertex) so that adjacent vertices are assigned a different color is a (legal) **coloring** of G.

TERMINOLOGY RELATED TO COLORINGS

- In a given coloring of a graph G, the set of all of those vertices assigned the same color is called a **color class**.
- A coloring of G produces a partition of $V(G)$ into different color classes, and each of these color classes is an independent set of vertices.
- A coloring that uses n colors is called a ***n*-coloring**.
- A graph whose vertices can be colored with n or fewer colors is called ***n*-colorable**.

CHROMATIC NUMBER

- The minimum number of colors in a coloring of G, where the minimum is taken over all colorings of G, is called the **chromatic number** of G and is denoted by $\chi(G)$.
- If G is a graph for which $\chi(G) = n$, then we say G is ***n*-chromatic**.

CHROMATIC NUMBER FOR SOME COMMON GRAPHS

- $\chi(C_{2p}) = 2$
- $\chi(C_{2p+1}) = 3$
- $\chi(K_p) = p$
- $\chi(K_{p_1,p_2,\ldots,p_n}) = n$
- In general, if G is a k-partite graph, $\chi(G) \leq k$.

n-CRITICAL AND *n*-MINIMAL GRAPHS

- A graph G is **critically *n*-chromatic**, or simply ***n*-critical** (if the context of coloring is clear) if $\chi(G) = n$ and $\chi(G - x) = n - 1$ for every $x \in V(G)$.
- A graph G is **minimally *n*-chromatic**, or simply ***n*-minimal** (if the context of coloring is clear) if $\chi(G) = n$ and $\chi(G - e) = n - 1$ for every $e \in E(G)$.

NOTE: Every graph contains an n-critical subgraph and an n-minimal subgraph. (Just remove vertices and/or edges until you reach the desired subgraph.)

MINIMUM DEGREE AND *n*-CRITICAL GRAPHS

Theorem 8.2.1: If G is a critically n-chromatic graph, then $\delta(G) \geq n - 1$.

Corollary 8.2.1:

1. Every n-chromatics graph has at least n vertices of degree at least $n - 1$.
2. For any graph G, $\chi(G) \leq \Delta(G) + 1$.

S-COMPONENTS

Let S be a vertex cut set in a connected graph G. Let the components of $G - S$ have vertex sets $V_1, V_2, \ldots, V_t$.

- The subgraphs $G_i = \langle V_i \cup S \rangle$ are called the ***S*-components** of G.
- Colorings of $G_1, G_2, \ldots, G_t$ **agree on *S*** if each vertex of S is assigned the same color in each of the colorings of the G_i $(i = 1,2, \ldots, t)$.

CONNECTEDNESS AND *n*-CRITICAL GRAPHS

Theorem 8.2.2: If G is a critically n-chromatic graph $(n \geq 4)$, then no vertex cut set induces a complete graph and, hence, G must be 2-connected.

Consequence: If an n-critical graph has a 2-vertex cut set $\{u, v\}$, then u and v cannot be adjacent.

EDGE CONNECTEDNESS AND n-CRITICAL GRAPHS

Theorem 8.2.3 (Dirac): Every critically n-chromatic graph ($n \geq 2$) is $n-1$ edge connected.

A COROLLARY OF THEOREM 8.2.1 AND 8.2.3

Corollary 8.2.2:

1. If G is a connected, n-minimal graph ($n \geq 2$), then G is $(n-1)$-edge connected.
2. If G is n-critical or connected and n-minimal, then $\delta(G) \geq n-1$.

COLOR-UNIQUE AND COLOR DISTINCT

- If an n-critical graph G has a two vertex cut set $\{u, v\}$, we know u and v cannot be adjacent.
- An $S = \{u, v\}$-component H of G is **color-unique** if every $(n-1)$-coloring of H assigns the same color to both u and v.
- An $S = \{u, v\}$-component H of G is **color-distinct** if every $(n-1)$-coloring of H assigns different colors to both u and v.

ANOTHER THEOREM OF DIRAC

Theorem 8.2.4 (Dirac): Let G be an n-critical graph with a two vertex cut set $S = \{u, v\}$. Then:

1. $G = H_1 \cup H_2$, where H_1 is a color-unique S-component and H_2 is a color-distinct S-component.
2. Both $H_1 + uv$ and the graph obtained from H_2 by identifying u and v are n-critical.

A COROLLARY

Corollary 8.2.3: Let G be an n-critical graph with a two vertex cut set $\{u, v\}$. Then

$$deg\, u + deg\, v \geq 3n - 5.$$

BROOKS' THEOREM

Theorem 8.2.5 (Brooks): If G is a connected graph that is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta(G)$.