Section 7.2

Matching Algorithms and Marriage

MAXIMUM MATCHING IN BIPARTITE GRAPH ALGORITHM

Algorithm 7.2.1 A Maximum Matching in a Bipartite Graph

Input: Let $G = (X \cup Y, E)$ be a bipartite graph and suppose that $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. Further, let M be any matching in G (including the empty matching).

Output: A matching larger than M or the information that the present matching is maximum.

Method: We now execute the following labeling steps until no step can be applied.

MAXIMUM MATCHING IN BIPARTITE GRAPH ALGORITHM (CONTINUED)

1. Label with an * all vertices of X that are weak with respect to M. Now, alternately apply steps 2 and 3 until no further labeling is possible.
2. Select a newly labeled vertex X, say x_i, and label with x_i all unlabeled vertices of Y that are joined to x_i by a weak edge with respect to M. Repeat this step on all vertices of X that were labeled in the previous step.
3. Select a newly labeled vertex of Y, say y_j, and label with y_j all the vertices of X that are joined to y_j by an edge in M. Repeat this process on all vertices in Y labeled in the previous step.

COMMENTS ON ALGORITHM 7.2.1

Notice the labelings will continue to alternate until one of two possibilities occurs:

$E1$: A weak vertex in Y has been labeled.

$E2$: It is not possible to label any more vertices and $E1$ has not occurred.

If ending $E1$ occurs, we have succeeded in finding an augmenting path. We can reconstruct this path by working backwards through the labels until we find the vertex of X which is labeled *.

The next theorem shows that if ending $E2$ occurs, M is already a maximum matching.

ALGORITHM 7.2.1 PRODUCES A MAXIMUM MATCHING

Theorem 7.2.1: Suppose that Algorithm 7.2.1 has halted with ending $E2$ occurring and having constructing a matching M. Let U_X be the unlabeled vertices of X and let L_Y be the labeled vertices of Y. Then $C = U_X \cup L_Y$ covers the edges of G, $|C| = |M|$, and M is a maximum matching in G.

JOB ASSIGNMENT PROBLEM REVISITED

Suppose now instead just having suitable candidates for jobs, we give each candidate an "unsuitability" rating for each job based on their qualifications (or lack thereof). The higher the number, the less qualified the applicant is for the job. This weights the edges in a bipartite graph. Obviously, we want the most qualified people in each job. So, we seek the matching with the minimum weight.

NOTE: We can assume the bipartite graph modeling the situation is complete since we can assign a weight of infinity (∞) to an edge for an applicant that is unqualified for a job.

WEIGHT OF A MATCHING

For any matching M, the **weight of the matching** is

$$W(M) = \sum_{e \in M} w(e) \ .$$

UNSUITABILITY MATRIX

If we have n applicants and n jobs and an unsuitability score for each applicant and job, we create an unsuitability matrix $U[w_{i\ k}]$ where $w_{i\ k}$ is the weight of the edge joining job j_i with applicant a_k.

AN ALGORITHM FOR FINDING A MINIMUM WEIGHT MATCHING

1. Subtract the minimum entry in each row from the entire row. If we have n zeros with no two in the same row or column, we have a minimum weight matching. If we do not, proceed to step 2.
2. Subtract the minimum entry in each column from the entire column. If we have n zeros with no two in the same row or column, we have a minimum weight matching. If we do not, proceed to step 3.

MINIMUM WEIGHT MATCHING ALGORITHM (CONCLUDED)

3. Draw lines through the rows and columns containing zeros using as few lines as possible to cover all the zeros. Then apply the following procedure.
   a) Let m be the smallest number that is not included in any of the crossed rows or columns.
   b) Subtract m from all uncrossed numbers.
   c) Leave numbers which are crossed once unchanged.
   d) Add m to all numbers which are crossed twice.

STABLE MATCHING ALGORITHM

Algorithm 7.23: Stable Matching Algorithm

Input: Given preferences tables for the men and women.

Output: A set of stable marriages.

1. Each man proposes to his first choice.
2. The women with two or more proposals respond by rejecting all but the most favorable offer. However, no woman accepts a proposal.
3. The men that were rejected propose to their next choice. Those that were not rejected continue their offers.
4. We repeat step 3 until we reach a stage where no proposal is rejected.

STABLE MARRIAGE THEOREM

Theorem 7.2.5: Given n men and n women, there always exists a set of stable marriages.