Section 7.1

Matchings and Bipartite Graphs

MATCHINGS

- Two distinct edges are **independent** if they are not adjacent.
- A set of pairwise independent edges is called a **matching**.
- In a graph G, a matching of maximum cardinality is called a **maximum matching** and its cardinality is denoted by $\beta_1(G)$.
- A matching that pairs all the vertices in a graph is called a **perfect matching**.

TERMINOLOGY

- An edge is said to be **weak with respect to a matching *M*** if it is not in the matching.
- An vertex is said to be **weak with respect to a matching *M*** if it is only adjacent to weak edges.
- An ***M*-alternating path** in a graph G is a path whose edges are alternately in a matching M and not in M (or conversely).
- An ***M*-augmenting path** is an alternating path whose end vertices are both weak with respect to M. Thus, an M-augmenting path both begins and ends with a weak edge.

A LEMMA

Lemma 7.1.1: Let M_1 and M_2 be two matchings in a graph G. Then each component of the spanning subgraph H with edge set

$$E(H) = (M_1 - M_2) \cup (M_2 - M_1)$$

is one of the following types:

1. An isolated vertex.
2. An even cycle with edges alternately in M_1 and M_2.
3. A path whose edges are alternately in M_1 and M_2 and such that each end vertex of the path is weak with respect to exactly one of M_1 and M_2.

BERGE'S CHARACTERIZATION OF MAXIMUM MATCHINGS

Theorem 7.1.1 (Berge): A matching M in a graph G is maximum if and only if there exists no M-augmenting path in G.

A SET MATCHED UNDER A MATCHING

Given a matching M, we say that a set S is **matched under M** if every vertex of S is incident to an edge in M.

HALL'S THEOREM

Theorem 7.1.2 (Hall): Let $G = (X \cup Y, E)$ be a bipartite graph. Then X can be matched to a subset of Y if and only if $|N(S)| \geq |S|$ for all subsets S of X.

Corollary 7.1.1: If G is a k-regular bipartite graph with $k > 0$, then G has a perfect matching.

SOME SET TERMINOLOGY

- Given sets $S_1, S_2, \ldots, S_k$, we say any element $x_i \in S_i$ is a **representative** for the set S_i which contains it.
- Given sets $S_1, S_2, \ldots, S_k$, a **system of distinct representatives** or a **traversal** of the sets is a set of representatives $x_i \in S_i$ such that $x_i \neq x_j$ whenever $i \neq j$.

SETS AND GRAPHS

- Let vertex s_i represent the set S_i.
- Use distinct vertices u_j to represent each element x_j in each of the sets.
- Join vertices s_i and u_j if and only if the element x_j is in set S_i.
- NOTE: $N(s_i) = \{u_j \mid x_j \in S_i\}$.
- Finding a system of distinct representative (SDR) is equivalent to finding a matching of the s_i's into a subset of the u_j's.

THE SDR THEOREM

<u>The SDR Theorem</u>: A collection $S_1, S_2, \ldots, S_k$, $k \geq 1$ of finite nonempty sets has a system of distinct representatives if and only if the union of any t of these sets contains at least t elements for each t, $(1 \leq t \leq k)$.

THE MARRIAGE THEOREM (A VERSION OF HALL'S THEOREM)

<u>The Marriage Theorem</u>: Given a set of n men and n women, let each man make a list of the women he is willing to marry. Then each man can be married to a woman on the list if and only if for every value of k $(1 \leq k \leq n)$, the union of any k of the lists contains exactly k names.

EDGE COVERS

- A set C of vertices is said to **<u>cover</u>** the edges of a graph G (or be an **<u>edge cover</u>**) if every edge in G is incident to a vertex of C.
- The minimum cardinality of an edge cover in G is denoted by $\alpha(G)$.

THE KÖNIG-EGERVÁRY THEOREM

Theorem 7.1.3 (König, Egerváry): If $G = (X \cup Y, E)$ is a bipartite graph, then the maximum number of edges in a matching in G equals the minimum number vertices in a cover for $E(G)$; that is, $\beta_1(G) = \alpha(G)$.

NETWORKS AND MATCHINGS

For a bipartite graph $G = (X \cup Y, E)$ we construct a network N_G corresponding to G as follows:

- Orient all edges of G from X to Y.
- Insert a source s with arcs to all vertices of X.
- Insert a sink t with arcs from all vertices of Y.
- Assign a capacity of 1 to all the arcs out of s or into t.
- Assign a capacity of ∞ to all arcs from X to Y.

MATCHINGS AND FLOWS

Theorem 7.1.4: In a bipartite graph $G = (X \cup Y, E)$, the number of edges in a maximum matching equals the maximum flow in the network N_G.