Section 6.3

A Planarity Algorithm

PIECES OF A GRAPH

- If $G_1 = (V_1, E_1)$ is a subgraph of $G = (V, E)$, then a **piece** of G relative to G_1 is either an edge $e = uv$ where $e \notin E_1$ and $u, v \in V(G_1)$ or a connected component of $(G - G_1)$ plus any edges incident to vertices of this component.
- For any piece P relative to G_1, the vertices of P in G_1 are called **contact vertices**.
- If a piece has two or more contact vertices, it is called a **segment**.
- Two segments are **incompatible** if when placed in the same region of the plane determined by a cycle C, at least two of their edges cross. Note that when embedded C divides the plane into two regions, one interior to C, the other exterior.

PRELIMINARY TEST TO SIMPLIFY FINDING A PLANAR EMBEDDING

1. If $|E| > 3p - 6$, then the graph must be nonplanar.
2. If the graph is disconnected, consider each component separately.
3. If the graph contains a cut vertex, then it is clearly planar if and only if each of the blocks is planar. Thus, we can limit our attention to 2-connected graphs.
4. Loops and multiple edges change nothing; hence, we need only consider graphs.
5. A vertex of degree 2 can certainly be replaced by an edge joining its neighbors. This **contraction** of all vertices of degree 2 constructs a homeomorphic graph with the smallest number of vertices. Certainly, a graph is planar if and only if the contraction is planar.

G-ADMISSIBLE

Let $\widehat{H}$ be a plane embedding of a subgraph H of G. If there exists a plane embedding of G (say $\widehat{G}$) such that $\widehat{H} \subseteq \widehat{G}$, then $\widehat{H}$ is said to be ***G*-admissible**.

SEGMENTS AND SUBGRAPHS

- Let S be any segment of G relative to a subgraph H. S can be drawn in region r of $\widehat{H}$ provided all the contact vertices of S lie in the boundary of r.
- We can extend the embedding of $\widehat{H}$ to include at least part of S.

STRATEGY OF THE DEMOUCRON, MALGRANGE, AND PERTUISET ALG.

- Find a sequence of subgraphs $\widehat{H}_1, \widehat{H}_2, \ldots, \widehat{H}_{|E|-p+2} = G$ such that $H_i \subset H_{i+1}$ and such that $\widehat{H}_i$ is G-admissible (if possible).
- We either construct a plane embedding of G (if one is possible) or discover some segment S which cannot be compatibly embedded in any region.

SOME NOTATION

Given a plane embedded subgraph $\widehat{H}_i$, for each segment S, the set $R(S, \widehat{H}_i)$ is defined to be the set regions in which S can be compatibly embedded in $\widehat{H}_i$.

DMP PLANARITY ALGORITHM

Algorithm 6.3.1 DMP Planarity Algorithm

Input: A preprocessed block (after applying tests 1-5).

Output: The fact that the graph is planar or nonplanar.

Method: Look for a sequence of admissible embeddings beginning with some cycle C.

DMP ALGORITHM (CONCLUDED)

1. Find a cycle C and a planar embedding of C as the first subgraph $\widehat{H}_1$. Set $i \leftarrow 1$ and $r \leftarrow 2$.
2. If $r = |E| - p + 2$,
   then stop;
   else determine all segments S of $\widehat{H}_i$ in G and for each segment S determine $R(S, \widehat{H}_i)$.
3. If there exists a segment S with $R(S, \widehat{H}_i) = \emptyset$,
   then stop and say G is nonplanar;
   else if there exists a segment S such that $|R(S, \widehat{H}_i)| = 1$,
   then let $\{R\} = R(S, \widehat{H}_i)$;
   else let S be any segment and R be any region in $R(S, \widehat{H}_i)$.
4. Choose a path P in S connecting two contact vertices. Set $H_{i+1} = H_1 \cup P$ to obtain the embedding $\widehat{H}_{i+1}$ with P placed in R.
5. Set $i \leftarrow i + 1$, $r \leftarrow r + 1$ and go to step 2.