Section 6.2

Characterizations of Planar Graphs

EDGE SUBDIVISION AND HOMEOMORPHIC

- By a **subdivision** of an edge $e = xy$, we mean that the edge xy is removed from the graph and a new vertex w is inserted in the graph along with the edges wx and wy.
- A graph H is **homeomorphic from** G if either H is isomorphic to G or H is isomorphic to a graph obtained by subdividing some sequence of edges of G.
- A graph G is **homeomorphic with** H if both G and H are homeomorphic from a graph F.
- "Homeomorphic with" is an equivalence relation.

SOME COMMENTS

- If a graph is planar, any graph obtained by subdividing edges is planer since all the added vertices have degree 2.
- If a graph is planar, then the graph obtained by **contracting** the vertices of degree 2 (replacing every vertex of degree 2 by an edge between its two neighbors) is also planar.
- Thus, a graph is planar if and only if all graphs homeomorphic with it are planar.

Currently, which graphs do we know are nonplanar?

- K_5
- $K_{3,3}$
- Graphs containing K_5 or $K_{3,3}$ as a subgraph.
- Graphs containing a subgraph homeomorphic with K_5 or $K_{3,3}$.

Kuratowski showed that up to homeomorphic graphs K_5 or $K_{3,3}$ are the only subgraphs that cause a graph to be nonplanar!!!!!

KURATOWSKI'S THEOREM

Theorem 6.2.1 (Kuratowski's Theorem): A graph G is planar if and only if G contains no subgraph homeomorphic with K_5 or $K_{3,3}$.

OUTLINE OF PROOF OF KURATOWSKI'S THEOREM

($\Rightarrow$) We have already discussed that if G is planar, it contains no subgraph that is a subdivision of $K_{3,3}$ or K_5.

Continued on next slide.

OUTLINE OF PROOF (CONTINUED)

($\Leftarrow$) Suppose G is a graph that contains no subdivision of $K_{3,3}$ or K_5. Here are the steps used to prove the result.

1. Prove that G is planar if and only if each block of G is planar.
2. Explain why it suffices to show that a block is planar if and only if it contains no subdivision of $K_{3,3}$ or K_5. Assume G is itself a nonplanar block of minimum size (connected with no cut vertex).

OUTLINE OF PROOF (CONTINUED)

3. Suppose that G is a nonplanar block that contains no subdivision of $K_{3,3}$ or K_5 (and search for contradiction).
4. Prove $\delta(G) \geq 3$.
5. Establish the existence of an edge $e = uv$ such that the graph $G - e$ is also a block.
6. Explain why $G - e$ is a planar graph containing a cycle C that includes both u and v, and choose C to have the maximum number of interior regions.

OUTLINE OF PROOF (CONCLUDED)

7. Establish several structural facts about the subgraphs inside and outside the cycle C.
8. Use these structural facts to demonstrate the existence of a subdivision of $K_{3,3}$ or K_5, thus establishing the contradiction (from step 3).