Section 5.2

Adjacency Conditions for Hamiltonian Graphs

HAMILTONIAN GRAPHS

- A graph is **hamiltonian** if it has a cycle containing all the vertices of the graph.
- The cycle itself is called an **hamiltonian cycle**.
- A path containing every vertex of a graph is called an **hamiltonian path**.

COMMENT

The problem of finding a hamiltonian cycle is NP-complete, and no practical characterization for hamiltonian graphs has been found.

ORE'S THEOREM

Theorem 5.2.1 (Ore): If G is a graph of order $p \geq 3$ such that for all pairs of distinct nonadjacent vertices x and y, $deg\ x + deg\ y \geq p$, then G is hamiltonian.

A COROLLARY

Corollary 5.2.1 (Dirac): If G is a graph of order $p \geq 3$ such that $\delta \geq \frac{p}{2}$, then G is hamiltonian.

NOTE: Dirac actually proved this before Ore proved Theorem 5.2.1. As a result, this corollary is sometimes called "Dirac's Theorem."

A COROLLARY

Corollary: If G is a graph of order $p \geq 3$ such that for all pairs of distinct nonadjacent vertices x and y, $deg\ x + deg\ y \geq p - 1$, then G contains a hamiltonian path.

A THEOREM OF BONDY AND CHVÁTAL

Theorem 5.2.2 (Bondy and Chvátal): Let x and y be nonadjacent vertices of a graph G of order p such that $deg\, x + deg\, y \geq p$. Then $G + xy$ is hamiltonian if and only if G is hamiltonian.

CLOSURE OF A GRAPH

The **closure** of a graph G, denoted by $CL(G)$, is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least p until no such pair remains.

CLOSURE IS WELL DEFINED

Theorem 5.2.3: If G_1 and G_2 are two graphs obtained by recursively joining pairs of nonadjacent vertices whose degree sum is at least p until no such pair remains, then $G_1 = G_2$; that is, $CL(G)$ is well defined.

THE CLOSURE OF A GRAPH AND HAMILTONIAN GRAPHS

Theorem 5.2.4: A graph is hamiltonian if and only if $CL(G)$ is hamiltonian.

ORE-TYPE GRAPHS

A graph G is said to be an **Ore-type** graph if $deg\, x + deg\, y \geq |V(G)|$ for all nonadjacent pairs of vertices x and y.

TWO OBSERVATIONS ABOUT ORE-TYPE GRAPHS

Observation 1: In any Ore-type graph G, the vertices of any maximal path can be permuted to form a cycle C.

Observation 2: In any Ore-type graph G, if C is any cycle (not necessarily hamiltonian) in G and $x \notin V(C)$, then there is a path that contains x and all the vertices of G.

ALBERTSON'S ALGORITHM

Algorithm 5.2.1 Albertson's Algorithm.

Input: An Ore-type graph G.

Output: A hamiltonian cycle in G.

Method: Use of observations 1 and 2.

ALBERTSON'S ALG. (CONCLUDED)

1. Create a maximal path $P: u, \dots, x_k, \dots, v$.
2. Repeat while $|V(P)| \neq |V(G)|$:
   If u is adjacent to v,
   then set $C: u, \dots, v, u$;
   else find a k such that u is adjacent to x_{k+1} and v is adjacent to x_k. Set $C = u, x_{k+1}, x_{k+2}, \dots, v, x_k, x_{k-1}, \dots, x_1, u$.
3. Find $x \in V(G - C)$, and create P^*, a path containing x and all of C. Set P equal to a maximal path containing P^*.
4. endwhile

SOME THEOREMS ON HAMILTONICITY

Theorem 5.2.5: If G is a 2-connected graph such that for every pair of nonadjacent vertices x and y,

$$|N(x) \cup N(y)| \geq \frac{2p-1}{3},$$

then G is hamiltonian.

Theorem 5.2.6: Let G be a k-connected graph of order $n \geq 3$. Suppose there exists some $t \leq k$ such that for every set S of t mutually nonadjacent vertices, $|N(S)| > \frac{t(n-1)}{t+1}$;

then G is hamiltonian.

SOME THEOREMS ON HAMILTONICITY (CONTINUED)

Theorem 5.2.7: If G is a 2-connected graph of sufficiently larger order n and if for each pair of arbitrary vertices $S = \{a, b\}$ we have that

$$|N(S)| \geq \frac{n}{2},$$

then G is hamiltonian.