Section 4.2

The Ford and Fulkerson Approach

AUGMENTING PATH

One approach to finding the maximum flow in a network N, is to start with some "path" between the source and sink and improve the flow along the path. Once this is done, we repeat the process on N with its modified flow. We continue until we cannot find a path whose flow can be approved. This is known as the **augmenting path** technique.

CUTS AND CAPACITY

- Let S be a subset set of V such that $s \in S$ and $t \in \bar{S} = V - S$.
- $out(S) = \{e = u \to v \in E : u \in S \text{ and } v \in \bar{S}\}$; that is, the set of all arcs from S to $\bar{S}$
- $in(S) = \{e = v \to u \in E : u \in S \text{ and } v \in \bar{S}\}$; that is, the set of all arcs from $\bar{S}$ to S
- $out(S) \cup in(S)$ is called the **cut** determined by S.
- For a set S of vertices, we call $c(S) = \sum_{e \in out(S)} c(e)$ the **capacity** of the cut determined by S.

SOME PREPARATORY RESULTS

Lemma 4.2.1: Given a network $N = (V, E, s, t, c)$ with flow f, then for every $S \subseteq V$ such that $s \in S$ and $t \in \bar{S}$,

$$F = \sum_{e \in out(S)} f(e) - \sum_{e \in in(S)} f(e).$$

Proposition 4.2.1: Given a network N, for every flow f with total flow F and for every $S \subseteq V$,

$$F \leq c(S).$$

Corollary 4.2.1: Given a network N with flow f and $S \subseteq V$ such that S contains s but not t, if $F = c(S)$, then F is a maximum and the cut determined by S is of minimum capacity.

SLACK AND SATURATION

- If the arc $e = x \rightarrow y$ is on an $s - t$ path and we wish use e to push more flow to t, then e presently must not be up to capacity; that is $f(e) < c(e)$. The amount of improvement is limited to $\Delta(e) = c(e) - f(e)$ called the **slack** of e.
- If $f(e) = c(e)$, we say the arc e is **saturated**.
- If the arc $e = y \rightarrow x$,then in order to increase the flow from s to t, we must cancel some of the flow into x on this arc (since the flow is away from t). Thus, there must already be some flow ($f(e) > 0$) on e if we are to increase the total flow.

AUGMENTING PATH

An **augmenting path** is a (not necessarily directed) path from s to t that can be used to increase the flow from s to t.

FORWARD AND BACKWARD LABELING

- In the augmenting path technique, we label vertices.
- A **forward labeling** of vertex v using arc $e = u \rightarrow v$ is done when u is labeled and v is not labeled and $c(e) > f(e)$. The label e^+ is assigned to v. Here $\Delta(e) = c(e) - f(e)$.
- A **backward labeling** of vertex v using arc $e = v \rightarrow u$ is done when u is labeled and v is not labeled and $f(e) > 0$. The label e^- is assigned to v. Here $\Delta(e) = f(e)$.

AUGMENTING PATHS AND MAXIMUM FLOW

Theorem 4.2.1: In a network N with flow f, the total flow F is maximum if and only if no augmenting path from s to t exists.

THE FORD AND FULKERSON ALGORITHM

Algorithm 4.2.1 The Ford and Fulkerson Algorithm.

Input: A network $N = (V, E, s, t, c)$ and a flow f. (Initially, we usually choose $f(e) = 0$ for every arc e.)

Output: A modified flow f^* or the answer that the present flow is maximum.

Method: Augmenting paths.

FORD AND FULKERSON ALG. (CONCLUDED)

1. Label s with $*$ and leave all other vertices unlabeled.
2. Find an augmenting path P from s to t.
3. If none exists,

   then halt, noting that the present flow is maximum;

   else compute and record the slack of each arc of P and compute the minimum slack λ. Now, redefine the flow f by adding λ to f for all forward arcs of P and subtracting λ from f for all backward arcs of P.
4. Set $f^* = f$ and repeat this process for N and the new flow f^*.

EDMONDS AND KARP TECHNIQUE

Edmonds and Karp were able to show that if a breadth-first search is used in the labeling algorithm and the shortest augmenting path is always selected, then the algorithm will terminate in at most $O(|V|^2|E|)$ steps even if irrational capacities are allowed.

SCAN

We will use the term **scan** to imply that a breadth-first search is being done.

FINDING AN AUGMENTING PATH ALGORITHM

Algorithm 4.2.2 Finding an Augmenting Path.

Input: A network N and a flow f.

Output: An augmenting path P or a message the none exists.

Method: A modified breadth-first search.

FINDING AN AUGMENTING PATH (CONCLUDED)

1. Label s with $*$.
2. If all labeled vertices have been scanned,

   then halt, noting that no augmenting path exists; hence the present flow is maximum,

   else find a labeled but unscanned vertex v and scan as follows: For each $e = vu \in out(b)$, if $c(e) - f(e) > 0$ and u is unlabeled, label u with e^+. For each $e = uv \in in(v)$, if $f(e) > 0$ and u is unlabled, then label u with e^-.
3. If t has been labeled,

   then starting at t, use the labels to backtrack to s along an augmenting path. The label at each vertex indicates its predecessor in the path. When you reach s out put the path and halt; else repeat step 2.

THE MAX FLOW-MIN CUT THEOREM

Theorem 4.2.2 (The Max Flow-Min Cut Theorem): In a network N, the maximum value of a flow equals the minimum capacity of a cut.