Section 3.3

Counting Trees

IDENTICAL GRAPHS

Two graphs G_1 and G_2 are **identical** if $V(G_1) = V(G_2)$ and $E(G_1) = E(G_2)$.

NUMBER OF NONIDENTICAL SPANNING TREES

Given a graph $G = (V, E)$ and let $V = \{1, 2, \dots, p\}$. How many nonidentical spanning trees are there?

CAYLEY'S TREE FORMULA

<u>Theorem 3.3.1 (Caley's Tree Formula)</u>: The number of nonidentical spanning trees on p distinct vertices is p^{p-2}.

PRÜFER'S TREE TO SEQUENCE ALGORITHM

Input: A tree T, with vertices labeled $1, 2, \ldots, n$.

1. Let $i \leftarrow 0$, and let $T_0 \leftarrow T$.
2. Find the leaf on T_i with the smallest label an call it v.
3. Record in the sequence σ the label of v's neighbor.
4. Remove v from T_i to create a new tree T_{i+1}.
5. If $T_{k+1} = K_2$, then halt. Otherwise, $i \leftarrow i + 1$ and go to step 2.

PRÜFER'S SEQUENCE TO TREE ALGORITHM

Input: A sequence $\sigma = a_1, a_2, \ldots, a_{p-2}$ of entries from the set $\{1, 2, \ldots, p\}$.

1. Draw p vertices and label them $1, 2, \ldots, p$. Let $S \leftarrow \{1, 2, \ldots, p\}$.
2. Let $i \leftarrow 0$, let $\sigma_0 \leftarrow \sigma$, and let $S_0 \leftarrow S$.
3. Let j be the smallest number in S_i that does not appear in the sequence σ_i.
4. Place an edge between vertex j and the vertex whose label appears first in the sequence σ_i.
5. Remove the first number in the sequence σ_i to create a new sequence σ_{i+1}. Let $S_{i+1} \leftarrow S - \{j\}$.
6. If the sequence σ_{i+1} is empty, place an edge between the two vertices whose labels are in S_{i+1}. Otherwise, $i \leftarrow i + 1$ and got to step 3.

DEGREE MATRIX

The $p \times p$ **degree matrix** $D = [d_{i\,j}]$ of a graph G is the matrix such that

$$d_{i\,j} = \begin{cases} \deg v_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

THE MATRIX-TREE THEOREM

Theorem 3.3.2 (The Matrix-Tree Theorem by Kirchhoff): Let G be a nontrivial graph with adjacency matrix A and degree matrix D. Then the number of nonidentical spanning trees of G is the value of any cofactor of $D - A$.