Section 1.5

Degree Sequences

DEGREE SEQUENCE

Given any graph G, it is easy to find the degree of each vertex. Each graph can be associated with a unique sequence called a **degree sequence**. The degree sequence is usually listed in nonincreasing sequence of nonnegative integers.

GRAPHICAL DEGREE SEQUENCE

- Given a sequence S, can we determine a graph with S as its degree sequence?
- Can we determine when a sequence of integers represents the degree sequence of a graph?
- A degree sequence is said to be **graphical** if it is the degree sequence of some graph.
- A graph G with degree sequence S is called a **realization** of S.

RESTRICTIONS FOR A GRAPHICAL DEGREE SEQUENCE

- All terms of the sequence must be nonnegative.
- If a degree sequence S has p terms, then no term can be larger than $p - 1$.
- The sum of the terms of a degree sequence must be even. That is, there must be an even number of odd terms in the sequence. (Theorem 1.1.1)

ARE REALIZATIONS UNIQUE?

If S is a graphical degree sequence and G is a realization of S, then G is not unique. That is, there can be more than one nonisomorphic graph that are realizations of S.

WHEN IS A DEGREE SEQUENCE GRAPHICAL

Theorem 1.5.1: A nonincreasing sequence of nonnegative integers

$$S: d_1, d_2, \dots, d_p \ \ (p \geq 2, d_1 \geq 0)$$

is graphical if and only if the sequence

$$S_1: d_2 - 1, d_3 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_p$$

is graphical.

EDGE INTERCHANGE

Let G be a graph and let x_i, x_j, x_k, x_l be vertices in G and $x_i x_j$ and $x_k x_l$ are edges in G. Further suppose x_i and x_k are not adjacent and neither are x_j and x_l. The deleting the edges $x_i x_j$ and $x_k x_l$ and inserting the edges $x_i x_k$ and $x_j x_l$ is called an **edge interchange**.

AN ALGORITHM TO TEST FOR A GRAPHICAL SEQUENCE

Algorithm 1.5.1 Test for a Graphical Sequence

Input: A sequence S of nonngative integers of length p.

Output: Yes if the sequence is graphical, no otherwise.

1. If there exists and integer d in S such that $d > p - 1$, then halt and answer no.
2. If the sequence is all zeros, then halt and answer is yes.
3. If the sequence contains a negative number, then halt and answer no.
4. Reorder the sequence (if necessary) so that it is nonincreasing.
5. Delete the first term d_1 from the sequence and subtract one from the next d_1 terms to form a new sequence. Go to step 2.

A THEOREM REALIZATIONS OF GRAPHICAL SEQUENCES

Theorem 1.5.2: Any realization of a graphical sequence can be obtained from any other realization by a finite number of edge interchanges.

AN ALTERNATE RESULT FOR TESTING GRAPHICAL SEQUENCES

Theorem 1.5.3: A nondecreasing sequence of nonnegative integers $S: d_1, d_2, \ldots, d_p$ $(p \geq 2)$ is graphical if and only if

$$\sum_{i=1}^{p} d_i$$

is even and for each integer k, $1 \leq k \leq p-1$,

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{p} \min\{k, d_i\}.$$

DEGREE SEQUENCES AND DIGRAPHS

A sequence $S: (i_1, o_1), (i_2, o_2), \ldots, (i_p, o_p)$ is called **digraphical** if it is the degree sequence of some digraph.

DIGRAPHICAL SEQUENCES

Theorem 1.5.4: A sequence $S: (i_1, o_1), (i_2, o_2), \ldots, (i_p, o_p)$ of ordered pairs of nonnegative integers with $i_1 \geq i_2 \geq \cdots \geq i_p$ is digraphical if and only if $i_k \leq p-1$ and $o_k \leq p-1$ for each k, and

$$\sum_{k=1}^{p} i_k = \sum_{k=1}^{p} o_k$$

and

$$\sum_{k=1}^{j} i_i \leq \sum_{k=1}^{j} \min\{j-1, o_k\} + \sum_{k=j+1}^{p} \min\{j, o_k\}$$

for $1 \leq j < p$.