Section 1.2

Elementary Properties and Operations

WALKS

- Let x and y be two vertices of a graph G (not necessarily distinct vertices). An $x - y$ **walk** in G is a finite alternating sequence of vertices and edges that begins with vertex x and ends with vertex y and in which each edge in the sequence joins the vertex that precedes it in the sequence to the vertex that follows it in the sequence.
- The number of edges is called the **length** of the walk.
- An $x - y$ walk is **closed** if $x = y$ and **open** otherwise.
- Two walks are equal if the sequences of vertices and edges are identical.
- Usually we will merely list the vertices in walk, noting that the edge between them (at least in a graph or digraph) is implied.

TRAILS AND PATHS

- An $x - y$ **trail** is an $x - y$ walk in which no edge is repeated.
- An $x - y$ **path** is an $x - y$ walk in which no vertex is repeated, except possibly for the first and last (if the path is closed).

RELATIONSHIP BETWEEN WALKS, TRAILS, AND PATHS

- Every path is a trail, and every trail is a walk. The converse of each of these is false.
- **Theorem 1.1.2:** In a graph G, every $x - y$ walk contains an $x - y$ path.

CIRCUITS AND CYCLES

- A closed trail is called a **circuit**.
- A closed path is called a **cycle**. (A cycle is also a circuit with no repeated vertices.)
- The **length** of a cycle (or circuit) is the number of edges in the cycle or circuit.

SOME SPECIAL GRAPHS

- A graph of order n consisting only of a cycle is denoted by C_n and is called an ***n*-cycle**.
- A graph of order n consisting only of a path is denoted by P_n and is called an ***n*-path**.
- We allow C_2 as a cycle, but note that it does not occur in graphs. We do ***not*** consider C_1 (a single vertex) as a trivial cycle.
- If a graph contains no cycles, it is termed **acyclic**.
- A graph of order p which contains an edge between all pairs of vertices is called a **complete graph** and is denoted by K_p.

CONNECTED GRAPHS

- We say a graph G is **connected** if there exists a path in G between any two of its vertices and G is **disconnected** otherwise.
- A **component** of a graph is maximal connected subgraph.
- NOTE: A is a **maximal subset** with property P if whenever B is another subset with property P and $A \subseteq B$ then $A = B$.

CONNECTED DIGRAPHS

- A digraph D is said to be **strongly connected** (or **strong**) if for each vertex v there exists a directed path from v to any other vertex.
- A digraph D is said to be **weakly connected** (or **weak**) if, when we remove the orientation from the arcs of D, a connected graph or multigraph remains.
- D is **disconnected** if it is not at least weakly connected.

TREES AND FORESTS

- A **tree** is a connected acyclic graph.
- A **forest** is an acyclic graph; that is, a graph each of whose components is a tree. (What else would a forest be?)

UNION OF GRAPHS

- The union of two graphs G_1 and G_2 (denoted by $G_1 \cup G_2$) is that graph G with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.
- The union of m isomorphic copies of the graph G is denoted by mG.

COMPLEMENT OF A GRAPH

The complement of a graph $G = (V, E)$ (denoted by $\bar{G}$) is the graph where $V(\bar{G}) = V(G)$ and $e \in E(\bar{G})$ if and only if e is not in $E(G)$.

THE JOIN OF TWO GRAPHS

- The **join** of two graphs G and H with disjoint vertex sets, denoted $G + H$, is the graph consisting of $G \cup H$ and all edges between vertices of G and H.
- If $H = K_1$ where K_1 is the single vertex x, we write the join as $G + x$.

PARTITE GRAPHS

- The **complete bipartite** graph, denoted by $K_{m,n}$ is the join of $\overline{K}_m$ and $\overline{K}_n$.
- A graph G is **bipartite** if it is possible to partition the vertex set V into two sets (called **partite sets**), say V_1 and V_2, such that each edge of G joins a vertex in V_1 with a vertex in V_2.
- A graph G is called ***n*-partite** if it is possible to partition the vertex set of G into n sets, such that any edge of G joins two vertices in different partite sets.

REMOVING VERTICES FROM A GRAPH

If we remove a set S of vertices from a graph G along with all the edges of G incident to a vertex in S, we denote the resulting graph by $G - S$. If $S = \{x\}$, we denote the resulting graph by $G - x$.

THE CARTESIAN PRODUCT OF GRAPHS

The **cartesian product** of graphs G_1 and G_2, denoted by $G_1 \times G_2$, is defined to be the graph with $V(G_1 \times G_2) = V(G_1) \times V(G_2)$, and two vertices $v = (v_1, v_2)$ and $w = (w_1, w_2)$ are adjacent in the cartesian product whenever $v_1 = w_1$ and v_2 is adjacent to w_2 in G_2 or symmetrically if $v_2 = w_2$ and v_1 is adjacent to w_1 in G_1.

THE LEXICOGRAPHIC PRODUCT OF TWO GRAPHS

The **lexicographic product** (sometimes called the **composition**) of two graphs G_1 and G_2, denoted by $G_1[G_2]$, has vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and (v_1, v_2) is adjacent to (w_1, w_2) if and only if either v_1 is adjacent to w_1 in G_1 or $v_1 = w_1$ in G_1 and $v_2 w_2 \in E(G_2)$.