Section 1.1

Fundamental Concepts and Notation

GRAPHS

A **graph** $G = (V, E)$ is a finite nonempty set V of elements called **vertices**, together with a set E of two element subsets of V called **edges**.

In our work, we will draw each vertex as a circle and each edge as a line joining two circles.

TERMINOLOGY

- Given a graph $G = (V, E)$. The number of vertices is called the **order of G** and the number of edges is called the **size of G**. We denote these by $|V|$ and $|E|$, respectively. If a graph has order p and size q, we say G is a (p, q) graph.
- Two vertices that are joined by an edge are said to be **adjacent**, as are two edges that meet at a vertex. If two vertices are not joined by an edge are called **nonadjacent** or **independent**. Two edges that do not share a common vertex are said to be **independent**.

MORE TERMINOLOGY

- The set of all vertices adjacent to a vertex v is called the **neighborhood of** v and is denoted by $N(v)$.
- An edge between vertices u and v is said to have u (or v) as an **end vertex**. The edge is said to be **incident** with u (or v), and v is said to **dominate** u (also, u dominates v).

DEGREE

- The number of edges incident with a vertex v is called the **degree of** v and is denoted by $deg\ v$ or $deg_G\ v$.
- The minimum and maximum degree of a vertex in the graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.
- A graph in which each vertex has degree r is called an ***r*-regular** graph (or simply regular).

THE FIRST THEOREM OF GRAPH THEORY

Theorem 1.1.1: Let G be a (p, q) graph and let $V = \{v_1, v_2, \ldots, v_p\}$. Then

$$\sum_{i=1}^{p} deg\ v_i = 2q$$

Consequently, any graph contains an even number of vertices of odd degree.

ISOMORPHIC GRAPHS

In mathematics, **isomorphic** means the "fundamental equality" of two objects or systems.

We say two graphs G_1 and G_2 are **isomorphic** if there exists a 1-1 and onto function $f: V(G_1) \to V(G_2)$ such that $xy \in E(G_1)$ if and only if $f(x)f(y) \in E(G_2)$ (that is, f preserves adjacency). The function f is called an **isomorphism**.

SUBGRAPHS

- A **subgraph** of G is any graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; we also say G **contains** H.
- If H is a subgraph of G and $V(H) = V(G)$, we say that H is a **spanning subgraph** of G.
- Given a subset S of $V(G)$, the **subgraph induced by S**, denoted $\langle S \rangle$, is the graph with vertex set S and edge set consisting of all of those edges of G incident with two vertices of S.

MULTIGRAPHS AND PSEUDOGRAPHS

- A **multigraph** is a graph with (possibly) multiple edges between vertices.
- A **pseudograph** allows edges that begin and end at the same vertex. Such edges are called **loops**.

DIRECTED GRAPHS

- If we think of the edge between two vertices as an ordered pair rather than a set, a natural direction from the first vertex in the pair to the second can be associated with the edge. Such an edge is called an **arc**.
- Graphs in which each edge has a direction are called **directed graphs** or **digraphs**.
- We will denote the arc directed from vertex u to vertex v as $u \rightarrow v$.
- If $u \rightarrow v$ is an arc of a digraph, we say that u **dominates** v or v is **dominated by** u. Sometimes we say u is **adjacent to** v or v is **adjacent from** u.

DEGREE IN DIRECTED GRAPHS

- The number of arcs directed away from a vertex v is called the **outdegree** of v, denoted by $od\ v$.
- The number of arcs directed into a vertex v is called the **indegree** of v, denoted by $id\ v$.
- In a digraph, we define the degree of the vertex v to be $deg\ v = id\ v + od\ v$.