Section 5.2
Section 5.3

Properties of Rational Functions;
The Graph of a Rational Function

RATIONAL FUNCTION

A **rational function** R is a function that can be written as

$$R(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomials.

NOTE: Polynomials are rational functions.

ASYMPTOTES

Many, though not all, rational functions have asymptotes. The common types of asymptotes are:

- **vertical asymptotes**
- **horizontal asymptotes**
- **oblique (slant) asymptotes**

FINDING THE DOMAIN OF A RATIONAL FUNCTION

The domain of a rational function is the set of all x-values that do ***not*** make the denominator zero.

CONTINUOUS FUNCTIONS

A function is **continuous** if "it can be drawn without lifting your pencil from your paper."

A place where you must lift your pencil is called a **discontinuity**.

For rational functions, discontinuities can be found by finding where the denominator is equal to zero.

LOCATING VERTICAL ASYMPTOTES

All vertical asymptotes occur at discontinuities.

Theorem: A rational function $R(x) = \frac{p(x)}{q(x)}$, ***in lowest terms***, will have a vertical asymptote $x = r$ if r is a real zero of the *denominator* q. That is, if $x - r$ is a factor of the denominator q of a rational function $R(x) = \frac{p(x)}{q(x)}$, in lowest terms, then R will have a vertical asymptote $x = r$.

NOTE: A graph can *never* cross a vertical asymptote.

MISSING POINT

A **missing point** is another type of discontinuity. A missing point is also called "a hole-in-the-graph."

Theorem: A rational function $R(x) = \frac{p(x)}{q(x)}$, that is ***not*** in lowest terms, will have a missing point at $x = r$ if r is a real zero of the denominator q but is *not* a real zero of denominator after R has been put into lowest terms.

HORIZONTAL ASYMPTOTES

- A horizontal asymptote is a type of end behavior for rational functions.
- The line $y = c$ is a horizontal asymptote of a rational function R if as the x-values get very small or very large, the y-values get close to c.
- NOTE: A graph may cross a horizontal asymptote.

OBLIQUE ASYMPTOTES

An **oblique** (or **slant**) **asymptote** is another type of end behavior for rational functions. Instead of the ends approaching a horizontal line, the ends approach a slanted line $y = mx + b$.

FINDING A HORIZONTAL OR OBLIQUE ASYMPTOTE

Let $R(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}$

be a rational function.

1. If $n < m$ (the degree of the numerator is less a than the degree of the denominator), the line $y = 0$ is a horizontal asymptote.
2. If $n = m$ (the degree of the numerator equals the degree of the denominator), the line $y = \frac{a_n}{b_m}$ is a horizontal asymptote. (That is the horizontal asymptote is equals the ratio of the leading coefficients.)
3. If $n = m + 1$ (the degree of the numerator is one more than the degree of the denominator), the line $y = mx + b$ is an oblique asymptote, which is the quotient found using long division.
4. If $n \geq m + 2$ (the degree of the numerator is two or more greater than the degree of the denominator), there is no horizontal or oblique asymptote.

FINDING THE OBLIQUE ASYMPTOTE

To find the oblique asymptote, use long division of polynomials. The oblique asymptote will be

$$y = \text{quotient}$$

A NOTE ABOUT HORIZONTAL AND OBLIQUE ASYMPTOTES

A rational function will never have both a horizontal asymptote and an oblique asymptote. A rational function may have neither a horizontal nor oblique asymptote.

ANALYZING THE GRAPH OF A RATIONAL FUNCTION

Step 1: Factor the numerator and denominator of R. Find the domain of the rational function.

Step 2: Write R in lowest terms.

Step 3: Find an plot any intercepts of the graph. Use multiplicity to determine the behavior of the graph of R at each x-intercept.

Step 4: Find any vertical asymptotes. Graph each vertical asymptote using a dashed line. Determine the behavior of the graph of R on either side of each vertical asymptote.

ANALYZING RATIONAL FUNCTIONS (CONCLUDED)

Step 5: Find the horizontal or oblique asymptote, if one exists. Find the points, if any, at which the graph of R intersects this asymptote. Graph the asymptote using a dashed line. Plot any points at which the graph of R intersects the asymptote.

Step 6: Using the zeros of the numerator and the denominator of R to the divide the x-axis into intervals. Determine where the graph of R is above or below the x-axis by choosing a number in each interval and evaluating R there.

Step 7: Use the results obtained in Steps 1 through 6 to graph R.